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Even for Nonpoint Events, Causality Implies the Lorentz Group

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There are many results that show that causality is indeed the fundamental notion of physics. In particular, it is known that for pointwise events, causality implies the Lorentz group. In quantum field theory, it is known that pointwise particles lead to divergences and therefore nonpoint particles are necessary (e.g., strings). Since particles are nonpoint, events occurring to these particles are also nonpoint events. In this paper, we show that even if we consider nonpoint events, causality still implies the Lorentz group. In other words, even for nonpoint events, the notion of causality is still fundamental.

1. INTRODUCTION

1.1. For Point Events, Causality Implies the Lorentz Group

One of the most fundamental physical notions is the notion of *causality*. Many notions can be described in terms of causality: e.g., from the causality relation of special relativity, we can uniquely determine the linear structure of space-time (Alexandrov, 1950; Alexandrov and Ovchinnikova, 1953; Zeeman, 1964; Kuz'minykh, 1975, 1976; Benz, 1977; Lester, 1977a, b; 1983; Kosheleva *et al.*, 1977a, b; Naber, 1992; Kreinovich, 1994).

The successful reformulation of different physical concepts in terms of the causality relation led many physicists to believe that causality is the "only physical variable" in the sense that everything else can be described in terms of it (see, e.g., Finkelstein and Gibbs, 1993; Finkelstein, 1996).

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In formal terms, the above result states the following:

Definition 1. By M, we will denote a 4-dimensional space R^4.

- Elements of the set M will be called *events*.
- We say that an event a *precedes* event b (or *causally precedes* b), and denote it by $a \leq b$, if

$$b_0 - a_0 \geq \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

Theorem (Alexandrov–Zeeman). Let f: $M \to M$ be a 1–1 mapping of M onto itself such that $a \leq b$ if and only if $f(a) \leq f(b)$. Then, f is linear. Moreover, f is a composition of a Lorentz transformation, a shift in 4D space-time, a 3D rotation, and a dilation.

1.2. In Quantum Field Theory, Events Are No Longer Pointwise

In the above result, we assumed that an event is a point in space-time, i.e., a moment of time in the life of a pointwise particle. In quantum field theory, it is known that pointwise particles lead to divergences; therefore, to avoid physically meaningless *infinities* and get physically meaningful *finite* values of physical quantities, we must consider nonpoint particles as well. For example, a consistent theory can be built on the assumption that particles are not *points*, but 1D *strings* in space.

When a *particle* is not necessarily a point in space, but may be a *set* of spatial points, an *event* in the life of a particle (e.g., the event of transforming one particle into another) is also no longer a point in space-time; it may correspond to a *set* of points in space-time.

So, in the quantum case, we get a set of events some of which are points in space-time and some of which are *sets* of points. From the mathematical viewpoint, points in space-time can be viewed as one-point sets, so we can simplify the mathematical picture by saying that *all* events are sets of points.

How can we define causality relation for these events? In general, the causality relation $a \leq b$ does not mean that the event a *necessarily* influenced the event b; it means simply that a *could* influence b. So, to define the causality relation between nonpoint events $A, B \subseteq M$, we must describe when an event A *could* influence the event B. Both events consist of several points in space-time, so A could influence B if and only if some point event comprising A could influence one of the point events which constitute the nonpoint event B. In mathematical terms, we thus say that $A \leq B$ if and only if $a \leq b$ for *some* $a \in A$ and $b \in B$.

Comment. The reader should be warned that although we use the *same symbol* $\leq$ to describe the old causality relation (between point events) and the new causality relation (between nonpoint events), some *properties* of the new causality relation are quite different from the properties of the old one. For example:

- For point events, causality is transitive: if $a \leq b$ and $b \leq c$, then $a \leq c$.
- However, the new relation $\leq$ is not always transitive: e.g., if for every real number r we denote $a_r = (r, 0, 0, 0)$ and take $A = \{a_2, a_3\}$, $B = \{a_1, a_2\}$, and $C = \{a_0, a_1\}$, then, as one can easily check, we have $A \leq B$, $B \leq C$, but $A \not\leq C$.

This nontransitivity may not sound so strange if we recall that $\leq$ means "can precede," i.e., "can influence". In the above example:

- A can influence B.
- B can influence C.
- But for A to be able to influence C we need *both* influences (A on B, and B on C), and although each of these influences can happen on its own, both of them cannot happen.

With this reformulation, the nontransitivity becomes no more surprising than, e.g., the fact that in traditional quantum mechanics:

- We can measure, with arbitrary accuracy, the *position* of a particle.
- We can measure, with arbitrary accuracy, this particle's *momentum*.
- But we cannot measure *both* position and momentum.

1.3. Formulation of the Problem

Now, comes the question: If we have the set of events (some pointwise, some nonpoint events), and if we only know the causality relation between these events, can we still reconstruct the linear structure of space-time?

In other words: *Will the notion of causality still be fundamental if we take nonpoint events into consideration*?

Our answer to this question is: *Yes, causality is still fundamental*.

2. DEFINITIONS AND THE MAIN RESULT

Definition 2. Let $M = R^4$.

- By a *general event*, we mean an arbitrary subset of the set M.
- We say that a general event A *can precede* a general event B, and denote it by $A \leq B$, if there exist points $a \in A$ and $b \in B$ for which $a \leq b$. The relation $\leq$ will be called the *causality relation* for general events.

- By a *set of events*, we mean a set E of general events that includes all one-point events.

Theorem. Let E be a set of events, and let F: $E \to E$ be a 1–1 mapping of the set E onto itself such that $A \leq B$ if and only if $F(A) \leq F(B)$. Then, there exists a linear mapping f: $M \to M$ such that for each one-point event $\{a\}$, $F(\{a\}) = \{f(a)\}$. Moreover, f is a composition of a Lorentz transformation, a shift in 4D space-time, a rotation in 3D space, and a dilation.

Comment. In other words, even for nonpoint events, causality still implies the Lorentz group.

3. PROOF

1. We start with a set of events E and a causality relation $\leq$ on this set. We do not know which of these events are one-point events, and which are not. Let us show that *we can determine whether an event A is a one-point event or not only by analyzing the causality relation.*

1.1. First, let us show that if $a \in A$, then for every event $B \in E$:

- If $B \leq \{a\}$, then $B \leq A$
- If $\{a\} \leq B$, then $A \leq B$

Without losing generality, let us prove the second implication (the first implication is proved similarly). By definition of the causality relation $\leq$ between events, $\{a\} \leq B$ means that $a' \leq b$ for some $a' \in \{a\}$ and $b \in B$. The only element a' in the set $\{a\}$ is the element a, so we can conclude that $a \leq b$ for some $b \in B$. Thus, $a \leq b$, where $a \in A$ and $b \in B$, which, by definition, means that $A \leq B$.

1.2. So, if A is not a one-point set, then there exists a set $A' \neq A$ with the following property.

(*) For every event B:

- If $B \leq A'$, then $B \leq A$
- If $A' \leq B$, then $A \leq B$

Indeed, according to point 1.1, as A' we can take a set $\{a\}$ for any $a \in A$.

1.3. Let us show that if A is a one-point set, i.e., if $A = \{a\}$ for some $a \in A$, then there cannot be a set $A' \neq A$ for which the property (*) is true. Indeed, let $A = \{a\}$ be a one-point set, and let A' be a set for which the condition (*) holds. Let us show that $A' = A$. Indeed, let a' be an arbitrary point from the set A'. Let us use the condition (*) for $B = \{a'\}$.

By definition of a causality relation between events, we have $B = \{a'\} \leq A'$ and therefore, due to condition (*), we have $B = \{a'\} \leq A = \{a\}$. By definition of $\leq$, this means that $a' \leq a$.

Similarly, from the second condition, we conclude that $a \leq a'$. Since a and a' are both points from M, from $a \leq a'$ and $a' \leq a$, we conclude that $a' = a$. Thus, every point $a' \in A'$ is equal to a, so $A' = \{a\} = A$. The statement is proven.

1.4. So, we can define a one-point set exclusively in terms of the causality relation $\leq$: an event A is a one-point set if and only if there does not exist another event $A' \neq A$ for which the condition (*) is true.

2. Since we have defined one-point sets exclusively in terms of the causality relation, any 1–1 mapping which preserves this relation therefore preserves the property "to be a one-point set." Thus, for every set $A = \{a\}$, the result $F(A)$ of the causality-preserving mapping F is also a one-point set, i.e., $F(A) = \{a'\}$ for some $a' \in M$. Let us denote the corresponding point a' by $f(a)$. Thus, f is a 1–1 mapping from M to M.

For one-point sets, the new causality relation $\{a\} \leq \{b\}$ coincides exactly with the old one $a \leq b$, so, from the fact that the mapping F preserves causality, we can conclude that the mapping f: $M \to M$ preserves causality as well. Thus, from the Alexandrov–Zeeman theorem, we can conclude that f is a linear mapping and, moreover, that f is a composition of a Lorentz transformation, a shift in 4-space, and a dilation.

The theorem is proven.

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